

Ch FRICTION.

Defⁿ: If two bodies be in contact with one another, the property of the two bodies by virtue of which a force is exerted b/w them at the pt of contact to prevent one body ^{from} sliding on the other, is called friction. Also The force exerted is called the force of friction.

→ 93, 98 '05 '09
H → Laws of statical friction:

Law I. When two bodies are in contact the direction of the friction on one of them at its pt of contact is opposite to the direction in which the pt of contact would commence to move.

Law II The magnitude of the friction is when there is equilibrium just sufficient to prevent the body from moving.

4/7 02 '05
4/7 2003 '06
Limiting friction:

Defⁿ: When one body is just on the pt of sliding upon another body, the equi-

Lebrium is said to be limiting and the friction then exerted is called limiting friction.

^{Nov 2021}
^{H - 98}
^{1/2020} Laws of limiting friction:

Law I The magnitude of the limiting friction bears a constant ratio to the normal reaction and this ratio depends on the substance of which the bodies are composed.

Law II The limiting friction is independent of the extent and shape of the surface in contact so long as the normal reaction is unaltered.

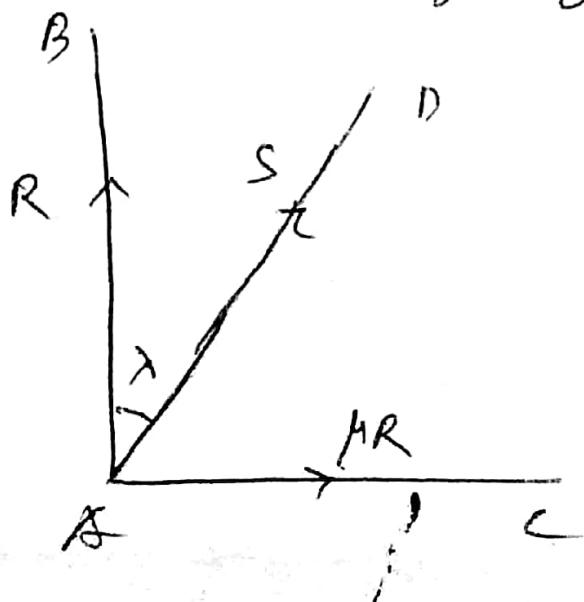
Law III When motion commences by one body sliding over the other, the direction of friction is opposite to the direction of motion; the magnitude of the friction is independent of the velocity but the ratio of the friction to the normal reaction is slightly less than when the body is at rest and is just on the eve of motion.

^{M-02}
^{Q-05}
^{Q-16}
^{A-10}
Co-efficient of friction: The constant ratio of the limiting friction to the normal reaction is called the co-efficient of friction generally denoted by μ ; hence if F be the friction and R the normal reaction we have when the equilibrium is limiting, $F = \mu R$.

^{G-95, 96, 09}
^{N-01, 01}
^{G-202}
¹⁰⁶
^{A-10}
Angle of friction:

When the equilibrium is limiting, if the force of friction and the normal reaction be compounded into one single force, the angle which this force makes with the normal is called the angle of friction.

Let A be the pt of contact of two bodies and let AB and AC be the directions



of the normal reaction R and friction μR .

Let AD be the direction of the resultant S , so that the angle of friction is $\angle BAD$. Let this angle be λ .

Since R and μR are the components of S , we have

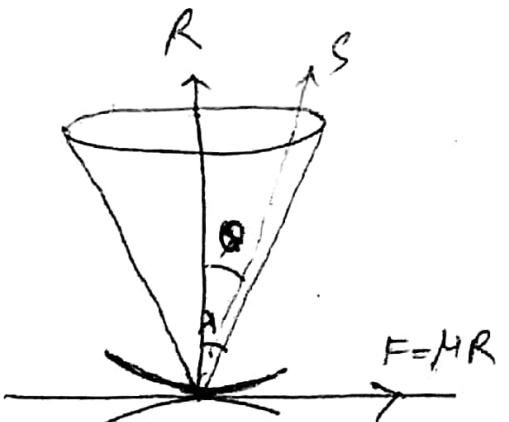
$$S \cos \lambda = R, S \sin \lambda = \mu R$$

$$\Rightarrow S = \sqrt{1 + \mu^2} \quad \& \tan \lambda = \mu$$

Thus the co-efficient of friction is equal to the tangent of the angle of friction.

~~M-93, 98
G-93, 98
H-93, 98~~
~~1900 or 1901~~
Cone of friction:

The inclination θ of the resultant reaction S to the normal reaction R increases with the force of friction F . When the equilibrium becomes limiting, F attains its



maximum value μR and θ becomes λ , the angle of friction.

It follows therefore, that equilibrium will be possible so long as θ remains less than λ and it will be limiting when θ equals λ , but if θ exceeds the value λ , motion must take place. As motion may ensue in all directions on the plane, the resultant reaction s will generate a cone.

Hence if we describe a cone with the pt of contact as the vertex and the common normal at the pt of contact as the axis and with semi vertical angle $\lambda (\leq \tan^{-1} \mu)$, then for equilibrium, the resultant reaction must lie within or upon the cone, but cannot have any direction lying outside the cone. This cone is called the cone of friction.

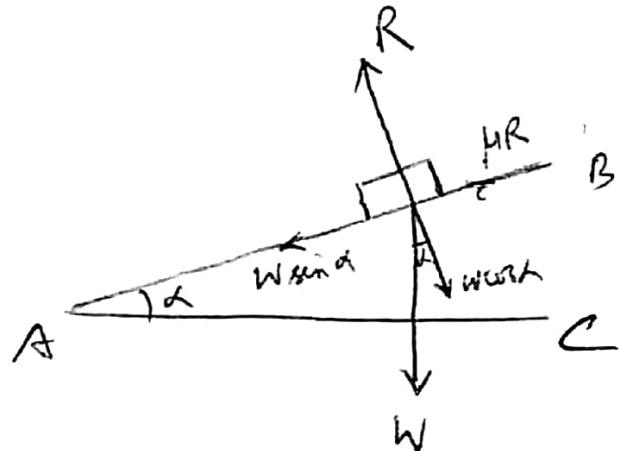
or.

A cone whose vertex is the pt of contact of two rough bodies and whose axis lies along the common normal and

whose semi-vertical angle is equal to angle λ i.e. the angle of friction is called the cone of friction.

Theorem: The greatest inclination of the rough inclined plane to the horizon at which the particle will remain at rest on it is equal to the angle of friction.

Pf. Let AB be the inclined plane inclined to the horizontal line AC at an angle λ .



Let w be the wt. of the body resting in limiting equilibrium on the rt of sliding down. Then the frictional force μR parallel to the plane is acting up the plane. R is the normal reaction and equal to $w \cos \lambda$. The downward force along the length of the plane acting

on the wt. is $w \sin \alpha$.

\therefore For equilibrium we have

$$\mu R = w \sin \alpha \text{ and } R = w \cos \alpha$$

$$\therefore \mu = \tan \alpha.$$

But if λ be the angle of friction, that is the angle between the normal to the surface and the resultant reaction, then

$$\tan \lambda = \frac{\mu R}{R} = \mu$$

$$\therefore \tan \alpha = \tan \lambda \text{ or } \alpha = \lambda.$$

If α is slightly greater, the downward force exceeds the frictional force and motion will ensue (begin).

Hence if α be the greatest inclination for equilibrium, α = the angle of friction.

$$4 \rightarrow 95$$

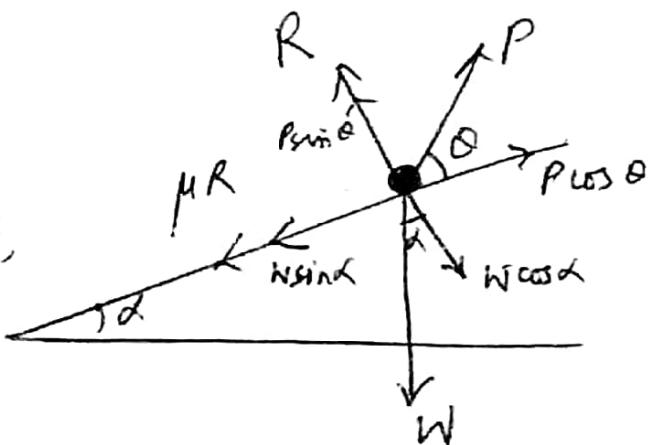
$$M \rightarrow 95$$

Σ least force required to pull a body up or down a rough inclined plane

Let w be the weight of the body placed on the rough inclined plane at O . Let R be the normal reaction and P be the force acting at an angle θ with the plane, which pulls the body.

G²⁰³
Motion up the plane: $\rightarrow w - P$

M²⁰²
 μ ²⁰⁵
 M ²⁰⁹ When the body is on the pt of moving up the plane, the force of friction μR will act down the plane.



Resolving the forces acting on the body along and perpendicular to the

inclined plane, we have

$$\mu R + w \sin \alpha = P \cos \alpha \quad \text{--- (i)}$$

$$\text{and } R + P \sin \alpha = w \cos \alpha \quad \text{--- (ii)}$$

From (i) & (ii) we get

$$\mu R = P \cos \alpha - w \sin \alpha$$

$$\text{and } R = w \cos \alpha - P \sin \alpha.$$

Eliminating R we get

$$\mu (w \cos \alpha - P \sin \alpha) = P \cos \alpha - w \sin \alpha$$

$$\Rightarrow P (\cos \alpha + \mu \sin \alpha) = w (\mu \cos \alpha + \sin \alpha)$$

$$\Rightarrow P \left(\cos \alpha + \frac{\sin \lambda}{\cos \lambda} \sin \alpha \right) = w \left(\frac{\sin \lambda}{\cos \lambda} \cos \alpha + \sin \alpha \right)$$

$\therefore \mu = \tan \lambda$

$$\Rightarrow P \cdot \frac{\cos \alpha \cos \lambda + \sin \alpha \sin \lambda}{\cos \lambda} = w \cdot \frac{\sin \lambda \cos \alpha + \cos \lambda \sin \alpha}{\cos \lambda}$$

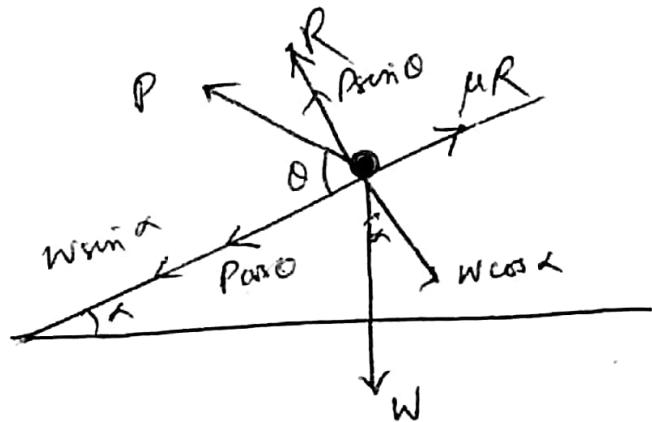
$$\Rightarrow P \cdot \cos(\alpha - \lambda) = w \sin(\lambda + \alpha)$$

$$\Rightarrow P = w \cdot \frac{\sin(\alpha + \lambda)}{\cos(\alpha - \lambda)}$$

$\therefore W, \alpha$ and λ are given, P is ^{the} least when $\cos(\theta - \lambda)$ is ^{the} greatest i.e. $\theta - \lambda = 0$ or $\theta = \lambda$ and then from (iii) the least value of P is $W \sin(\lambda + \lambda)$.

~~$\alpha^2 \theta^3$~~ Motion down the plane:

When the body is on the st of moving down the plane, the force of friction μR will act up the plane.



Resolving the forces acting on the body along and perpendicular to the inclined plane, we have

$$\mu R = W \sin \alpha + P \sin \theta \quad \dots (i)$$

$$\& R = P \cos \theta = W \cos \alpha$$

$$\text{or } R = W \cos \alpha - P \sin \theta \quad \dots (ii)$$

Eliminating R between (i) & (ii), we

$$\text{Hence } \mu(w \cos \alpha - P \sin \theta) = w \sin \lambda + P \cos \lambda$$

$$\text{or } P(\cos \theta + \mu \sin \alpha) = w(\mu \cos \alpha - \sin \lambda)$$

$$\text{or } P \left(\cos \theta + \frac{\sin \lambda}{\cos \lambda} \cdot \sin \alpha \right) = w \left(\frac{\sin \lambda}{\cos \lambda} \cdot \cos \alpha - \sin \lambda \right)$$

$$\text{or } P \cos(\theta - \lambda) = w \sin(\lambda - \alpha)$$

$$\Rightarrow P = \frac{w \sin(\lambda - \alpha)}{\cos(\theta - \lambda)}$$

\therefore When w , λ and α are given, P is least when $\cos(\theta - \lambda)$ is greatest i.e. when $\theta - \lambda = 0$ or $\theta = \lambda$ and then from (iii) the least value of P is $w \sin(\lambda - \alpha)$ where $\alpha < \lambda$.

Note 1. In case $\alpha > \lambda$, The body will move down the plane under its own weight and normal reaction.

Note 2. In case $\alpha = \lambda$, The body will be in limiting equilibrium i.e. on the brt of motion down the plane.

Ex. \rightarrow 88, 97 or
~~Text~~
~~M-15~~
A uniform ladder is in equilibrium
with one end on the ground and the other

against a vertical wall. To the ground and the wall be both rough the co-efficient of friction being μ and μ' respectively, and if the ladder be

is

$$\tan^{-1} \left(\frac{1 - \mu\mu'}{2\mu} \right).$$

Soln.

Let AB be the ladder and let its length be $2a$. Let R and S be the normal reactions on the ground and the wall, then the

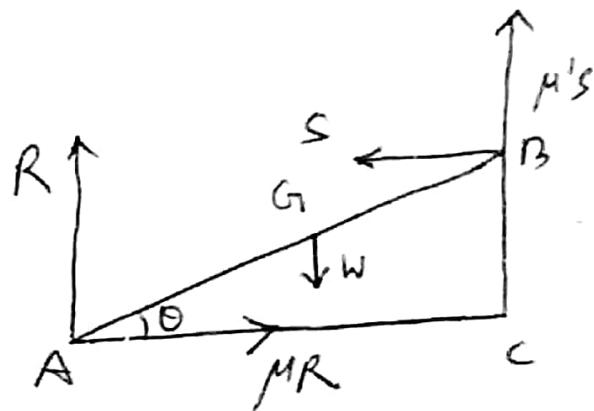
limiting friction at these points are μR and $\mu' S$ in directions shown in the figure.

Now for equilibrium, resolving horizontally and vertically and taking moments about A, we get

$$S = \mu R \quad \dots \dots (i)$$

$$R + \mu' S = W \quad \dots \dots (ii)$$

where W is the weight of the ladder acting



at G and where $BC_2 = AG = a$.

Also \rightarrow taking moment about A,

$$S. 2a \sin \theta + \mu' S. 2a \cos \theta - W \cdot a \cos \theta = 0 \quad \dots \text{(ii)}$$

From (i) & (ii), $S\left(\frac{1}{\mu} + \mu'\right) = W$ and then
from (iii),

$$2S(\tan \theta + \mu') = W = S\left(\frac{1}{\mu} + \mu'\right)$$

$$\therefore \tan \theta = \frac{1}{2} \left(\frac{1}{\mu} + \mu' \right) - \mu'$$

$$= \frac{1 - \mu \mu'}{2\mu}$$

$$\Rightarrow \theta = \tan^{-1} \left(\frac{1 - \mu \mu'}{2\mu} \right). //$$

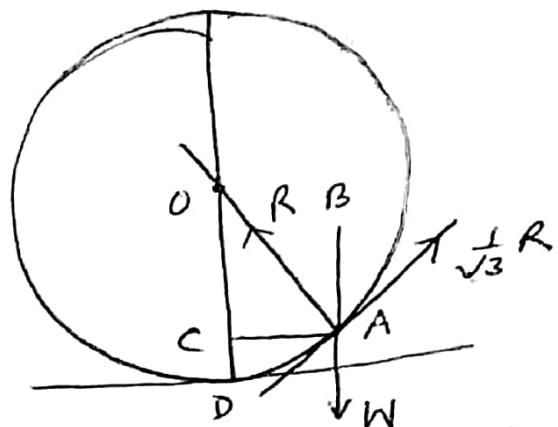
Ex. G-87, 90, 92, 98

~~G-87/09~~ (a) How high a particle can rest
inside a hollow sphere of radius 'a',
co-efficient of friction is $\sqrt{3}$.

\downarrow G-87

~~G-87~~ (b) If μ is the co-efficient of friction,
prove that the greatest distance from the
vertical diameter is $\frac{\mu a}{\sqrt{1 + \mu^2}}$.

Soln: Let A be the position of the particle. Then its height is CD along vertical. The particle is in equilibrium under the action of the forces, its own wt, normal reaction R along AO and the force of friction f_R along the tangent as shown in figure.



The resultant reaction must be along AB as $AB \perp$ vertical.
Hence $\angle BAO$ is the angle of friction, let it be λ .

$$\text{Then } \tan \lambda = \frac{1}{\sqrt{3}} \quad \therefore \lambda = 30^\circ.$$

$$\therefore \angle AOC = 30^\circ.$$

Now from $\triangle AOC$,

$$OC = AO \cos \angle AOC = a \cos 30^\circ = a \cdot \frac{\sqrt{3}}{2}.$$

$$\therefore CD = OD - OC = a - a \frac{\sqrt{3}}{2} = a \left(1 - \frac{1.732}{2}\right) \\ = a (1 - 0.866) = 0.134 a.$$

Hence the particle may be placed at a height $0.134 a$, above the horizontal plane through D.

(6) The greatest distance from the vertical diameter, i.e.

$$CA = OA \sin \theta \quad \text{where } \angle AOC = \theta$$

$$= a \cdot \frac{\mu}{\sqrt{1+\mu^2}} ; \text{ where } \tan \theta = \mu$$

\rightarrow 2022 (AB = 1, AG = a).

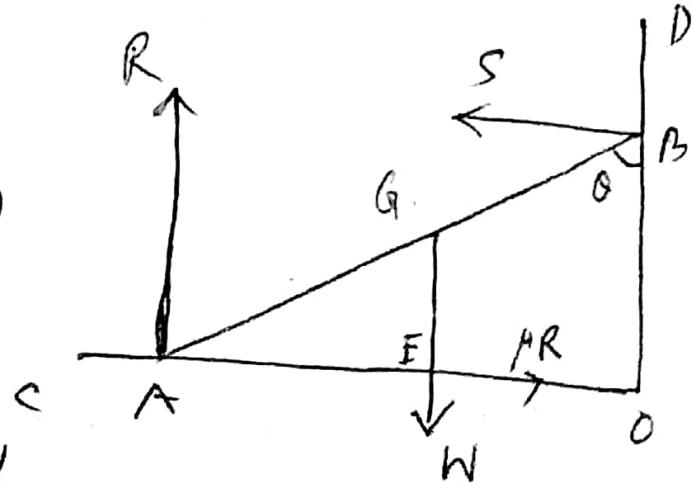
G → 86, 95

Ex- A uniform ladder rests in limiting equilibrium with one end on a rough floor, whose co-efficient of friction is μ and with the other against a smooth vertical wall; show that its inclination to the vertical is $\tan^{-1}(2\mu)$.

Sol: Let AB be

the ladder with
it end A resting
on the rough (floor)
horizontal base and
it end B against
the smooth vertical
wall OTBD.

$$\angle AOB = 90^\circ$$



Let the reaction at A be R . This will act perpendicular to the horizon OAE at A. Since the rod is in equilibrium, maximum friction at A will be in action. As the end A of the rod has a tendency to move towards C, this maximum friction which is equal to μR acts towards O along AO. The reaction S of the smooth wall at B is perpendicular to the wall. Hence it is horizontal.

Lastly the wt. of the rod is acting vertically downwards through the mid-point G of the rod. Thus the rod is in equilibrium under the action of four forces, namely the two reactions at A and B, its own wt. w and the force of friction μR .

Let $\angle ABO = \theta$. Resolving horizontally and vertically, we get—

$$S = \mu R \quad \& \quad R = w.$$

Taking moments about A, we get—

$$S \cdot OB = w \cdot AE \quad \text{or} \quad S \cdot AB \cos \theta = w \cdot AE \sin \theta$$

$$\text{or } \mu R \cdot AB \cos \theta = R \cdot \frac{AB}{2} \sin \theta / \cancel{AB} \quad \begin{array}{l} \cancel{2000N} \\ AB = l, AG = a \end{array}$$

$$\Rightarrow \tan \theta = 2\mu$$

$$\Rightarrow \theta = \tan^{-1}(2\mu)$$

$\cancel{\bullet} \Rightarrow \mu \cdot l \text{ will be constant}$
 $\Rightarrow \tan \theta = \frac{\mu l}{a}$
 $\text{but } \theta = \tan^{-1}\left(\frac{\mu l}{a}\right)$

G → 87
M → 86